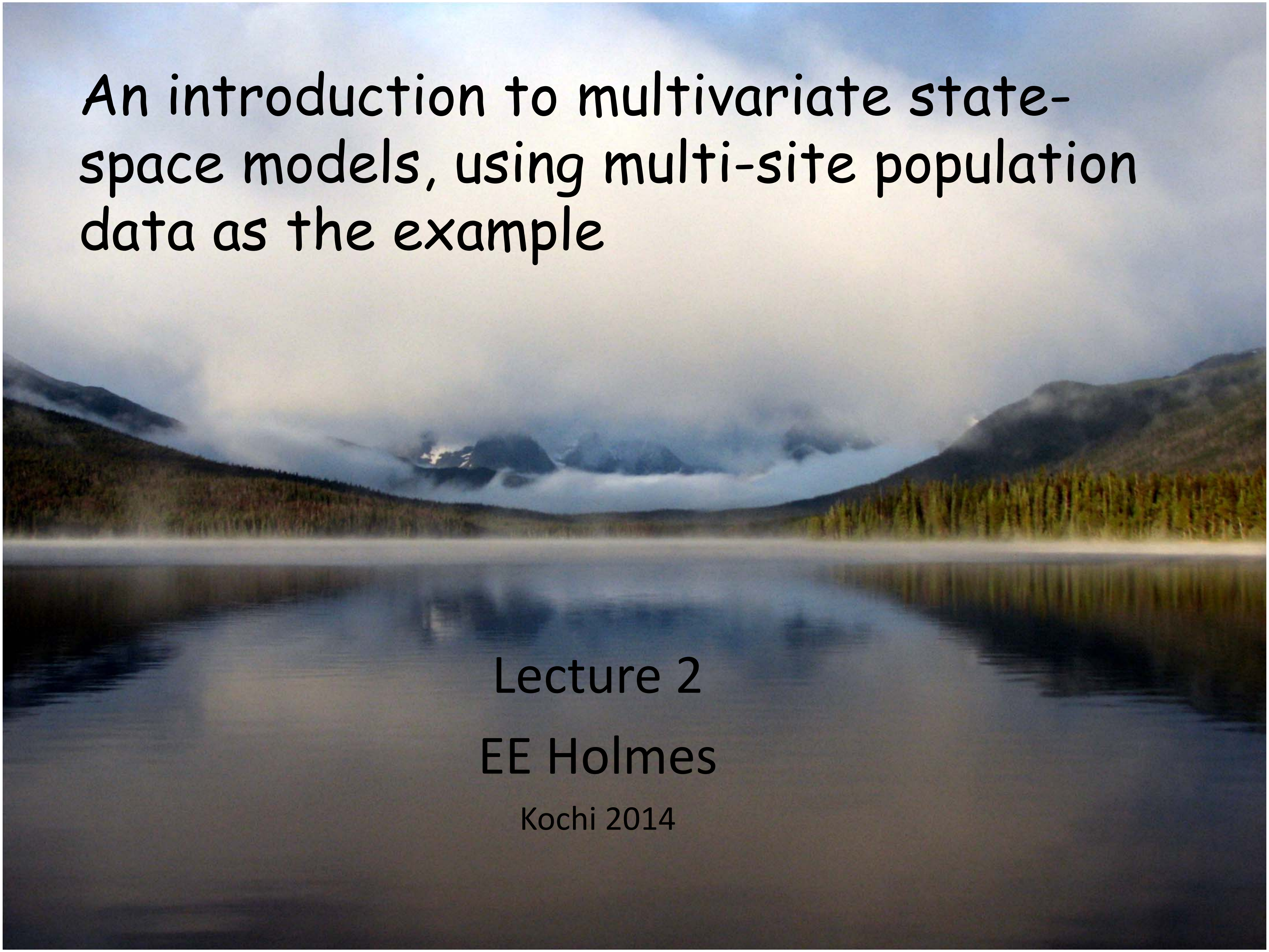


An introduction to multivariate state-space models, using multi-site population data as the example

Lecture 2

EE Holmes

Kochi 2014

Topics

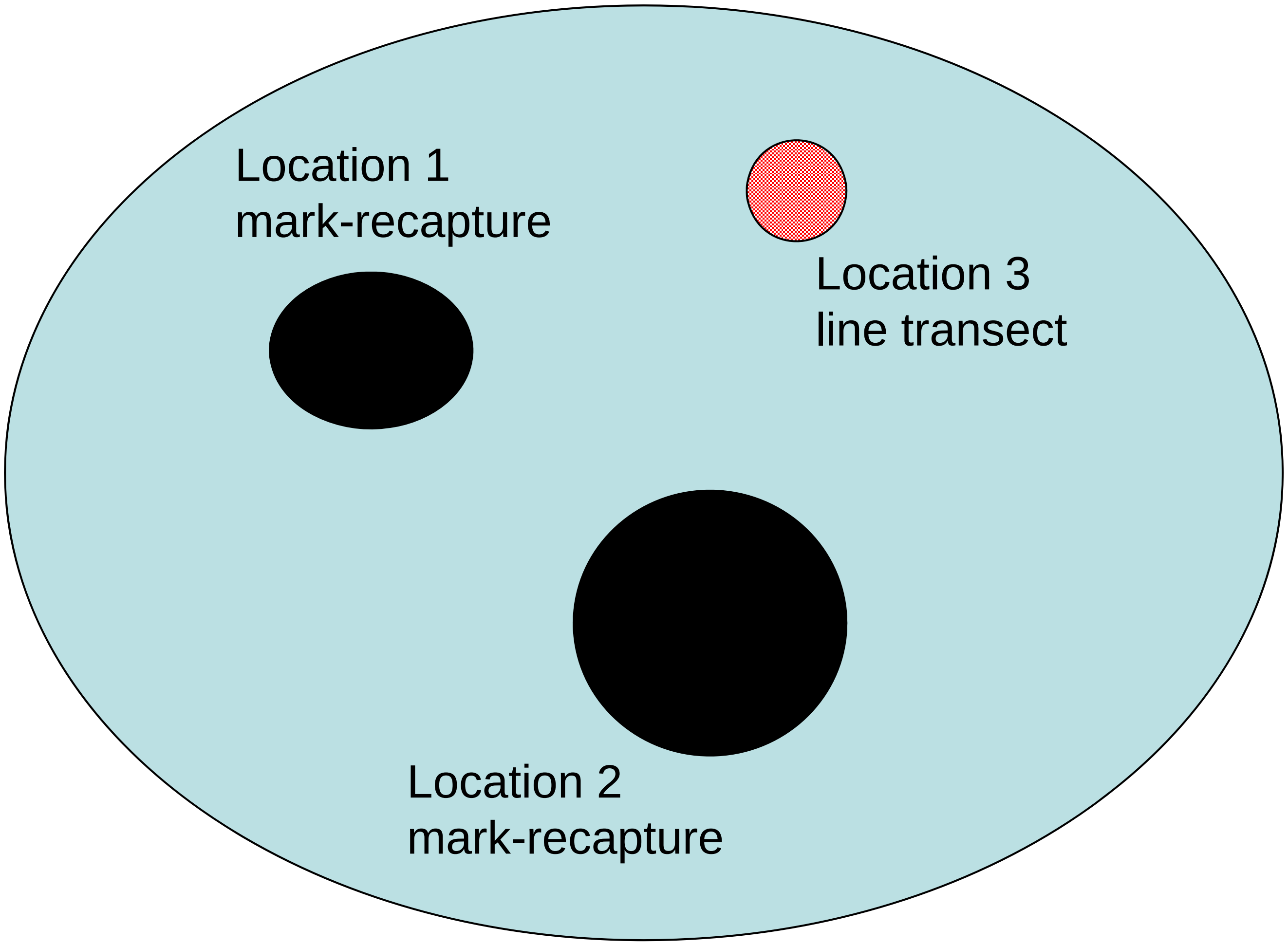
Lecture

- Short example of multivariate observations
- Examples of multivariate structure in population data
- How to express these structures mathematically
- Adding a multivariate observation process
- Model comparison using AIC and AIC weights

Computer Labs

- Analysis of population structure using multi-site data
- Combining diverse data sources to estimate an underlying model

Imagine we have 3 sampling locations for a population (denoted x).



Mathematically we can express this like

$$x_t = x_{t-1} + u + w_t, w_t \sim N(0,q)$$

$$y_{1,t} = x_t + v_{1,t}, v_{1,t} \sim N(a_1, r_1)$$

$$y_{2,t} = x_t + v_{2,t}, v_{2,t} \sim N(a_2, r_2)$$

$$y_{3,t} = x_t + v_{3,t}, v_{3,t} \sim N(a_3, r_3)$$

observations

population
size

noise

The observation part can be rewritten as a matrix

We need to fix one of the a's.
Traditionally we fix to 0.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_t = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t$$

observations Z matrix population size bias noise

The model with one a fixed to zero

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_t = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t$$

observations   Z matrix   population size   bias   noise

The observation errors are multivariate normal.

The variance-covariance matrix tells you how the observation errors are related. Are they independent? Or do they covary? Do have the same variance or difference variances?

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}_t \sim MVN\left(0, \begin{bmatrix} \eta_1^2 & \eta_{1,2} & \eta_{1,3} \\ \eta_{1,2} & \eta_2^2 & \eta_{3,2} \\ \eta_{1,3} & \eta_{3,2} & \eta_3^2 \end{bmatrix} \right)$$

The observation errors have a var-cov matrix

$$\begin{bmatrix} \eta_1^2 & \eta_{1,2} & \eta_{1,3} \\ \eta_{1,2} & \eta_2^2 & \eta_{3,2} \\ \eta_{1,3} & \eta_{3,2} & \eta_3^2 \end{bmatrix}$$

unconstrained

$$\begin{bmatrix} \eta^2 & \alpha & \alpha \\ \alpha & \eta^2 & \alpha \\ \alpha & \alpha & \eta^2 \end{bmatrix}$$

"equal varcov"

$$\begin{bmatrix} \eta_1^2 & 0 & 0 \\ 0 & \eta_2^2 & 0 \\ 0 & 0 & \eta_3^2 \end{bmatrix}$$

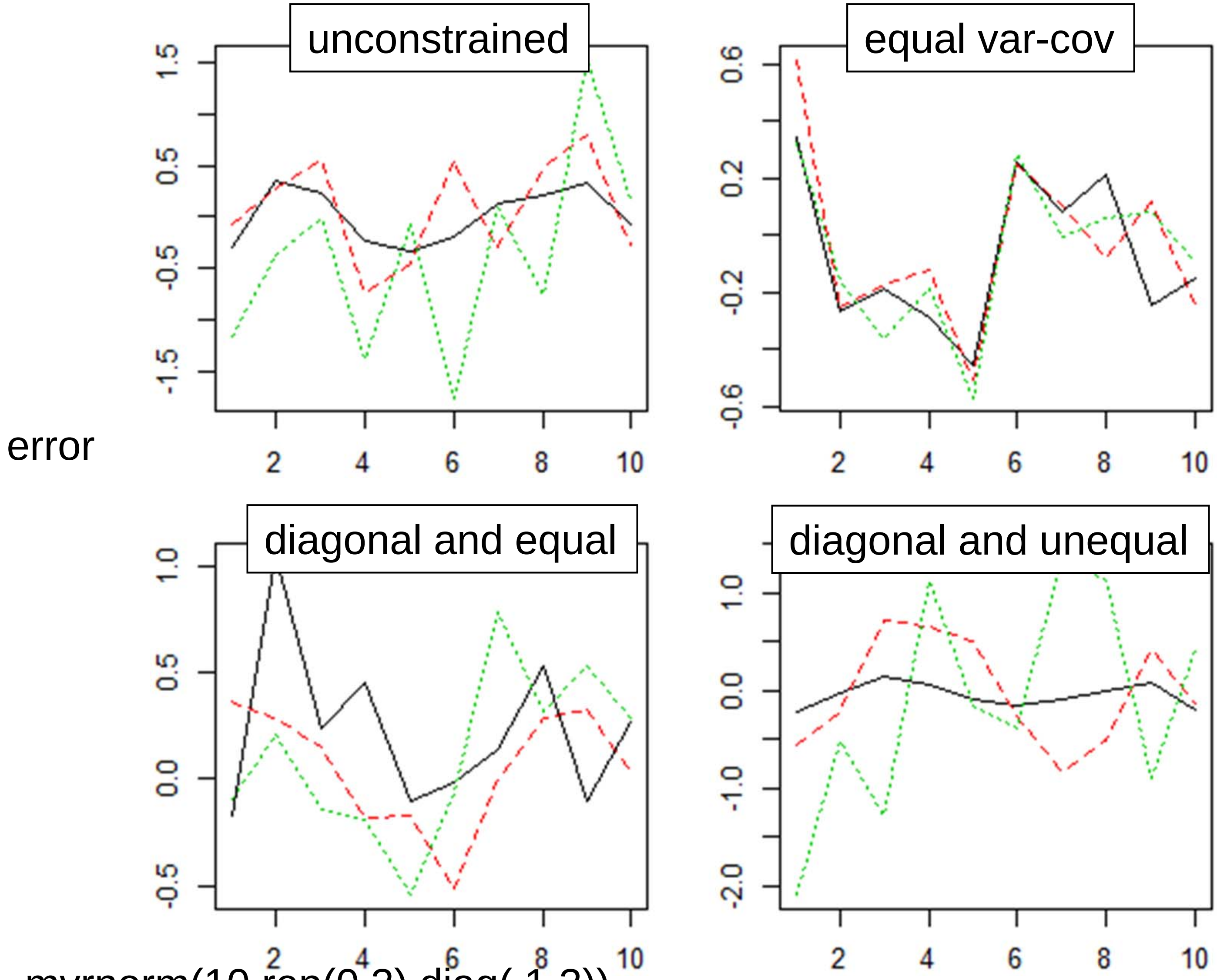
unique variances and uncorrelated errors

diagonal

$$\begin{bmatrix} \eta^2 & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^2 \end{bmatrix}$$

identical variances and uncorrelated errors

Example of errors coming from these variance-covariance matrices

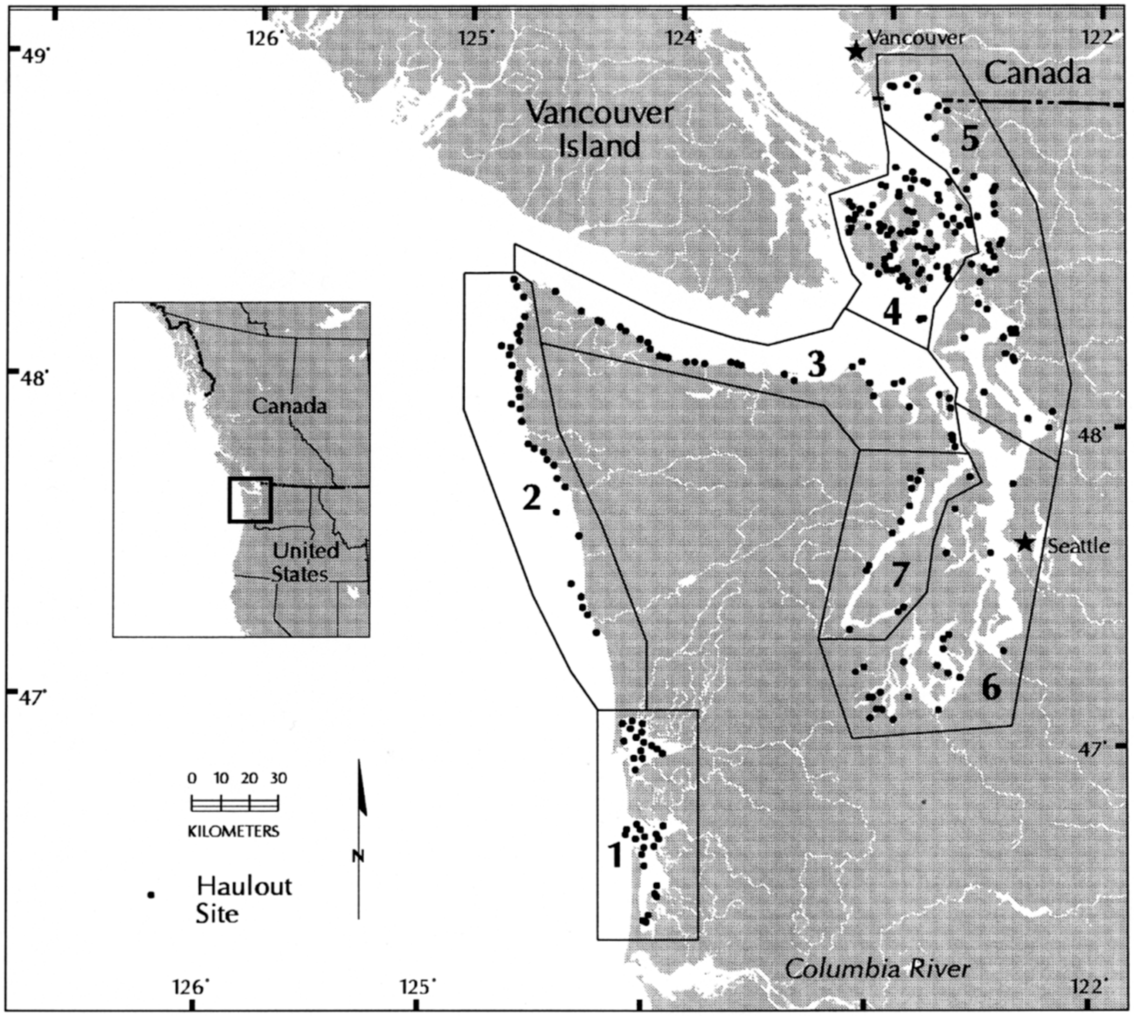


mvrnorm(10,rep(0,3),diag(.1,3))

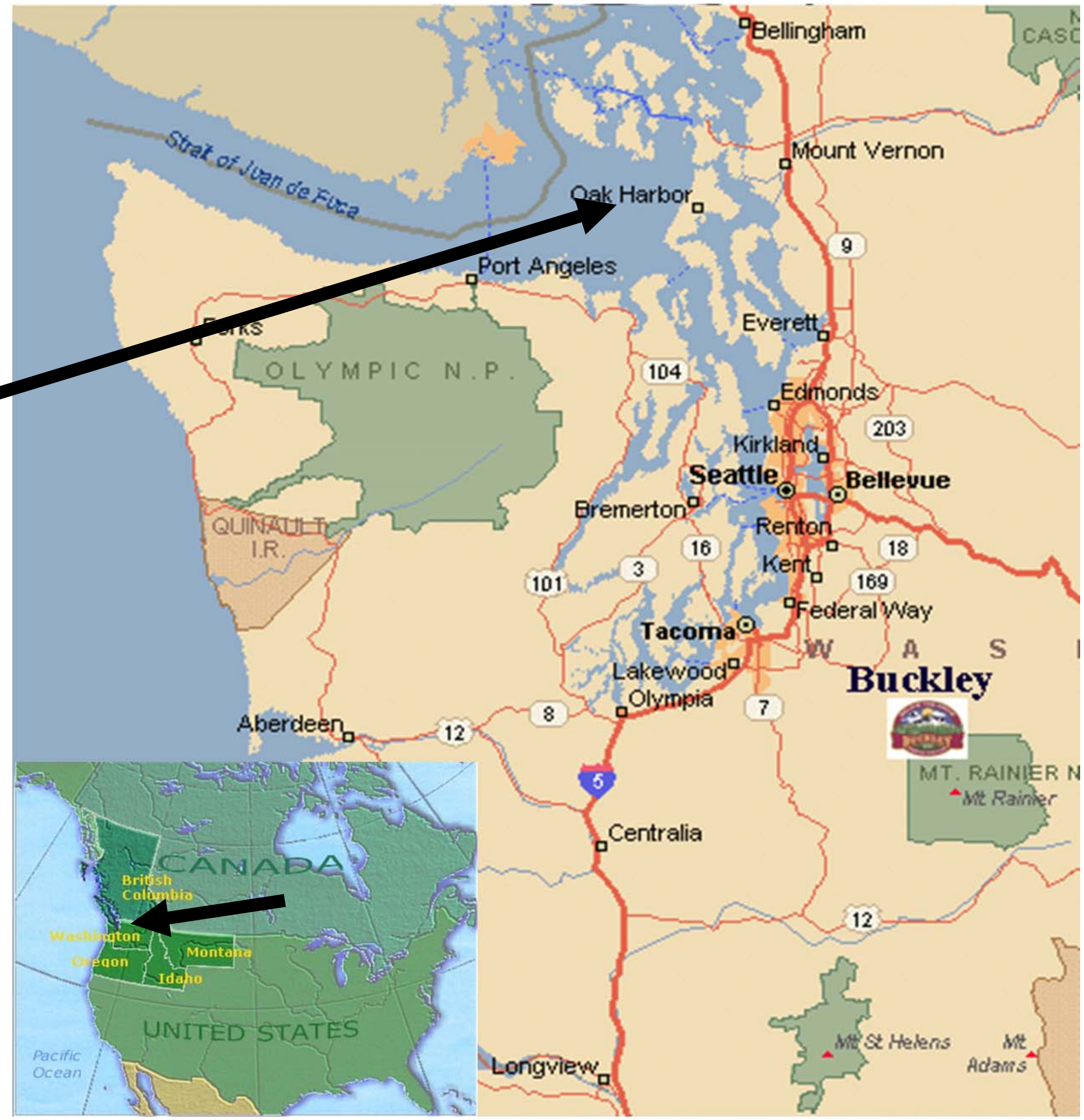
Some short examples. Estimates x with 3 observation time series

- lecture_3_example_1.R
- lecture_3_example_2.R
- lecture_3_example_3.R

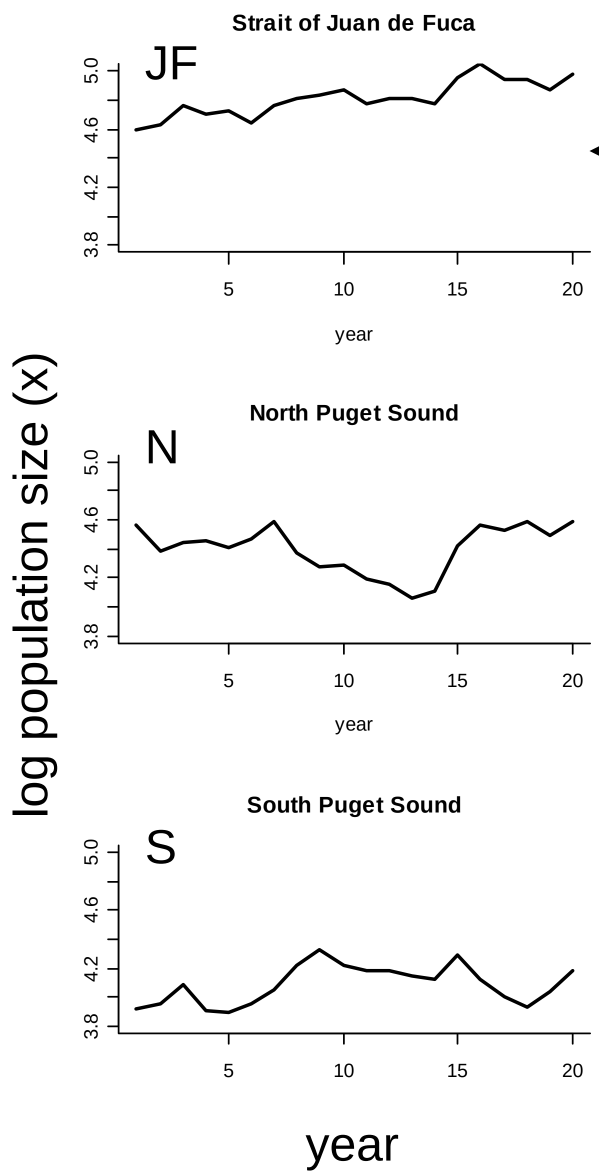
Multi-site data (Pacific harbor seals)

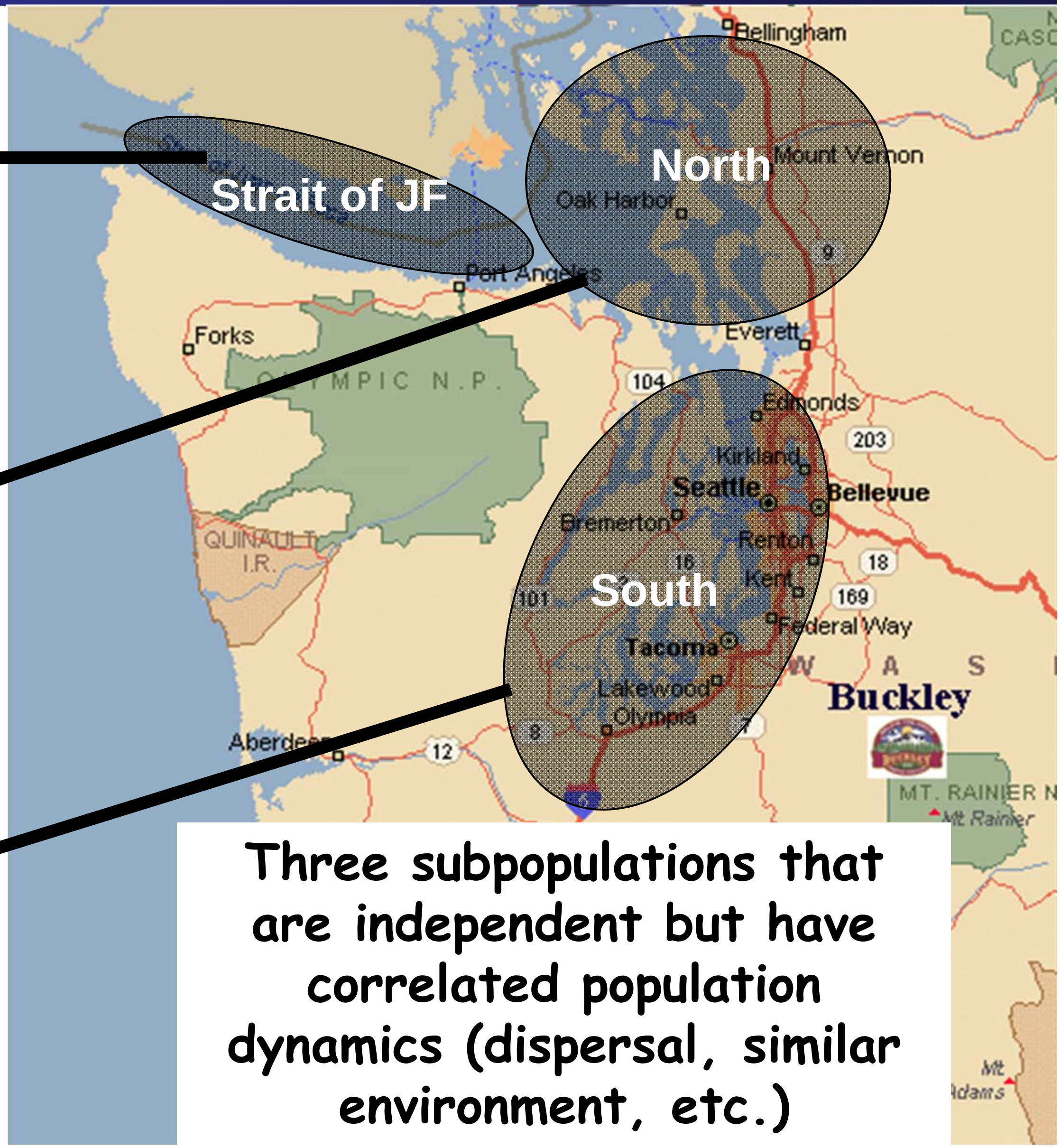


An example: modeling the population dynamics of harbor seals in Puget Sound, WA



Let's hypothesize (and model) that the population has 3 subpopulations





Three subpopulations that are independent but have correlated population dynamics (dispersal, similar environment, etc.)

A multivariate model for 3 subpopulations

Multivariate stochastic exponential growth

$$\begin{bmatrix} x_{JF,t} \\ x_{N,t} \\ x_{S,t} \end{bmatrix} = \begin{bmatrix} x_{JF,t-1} \\ x_{N,t-1} \\ x_{S,t-1} \end{bmatrix} + \begin{bmatrix} u_{JF} \\ u_{N} \\ u_{S} \end{bmatrix} + \begin{bmatrix} e_{JF,t} \\ e_{N,t} \\ e_{S,t} \end{bmatrix}$$

3 different x's, one for each subpopulation

3 mean population growth rate terms

3 different process errors

$\boldsymbol{e} \sim \mathrm{MVN}(0,\mathbf{Q})$

The population model in matrix notation

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{u} + \mathbf{w}_t$$

$$\mathbf{w}_t \sim \mathrm{MVN}(0,\mathbf{Q})$$

Each parameter has "structure". Different structures imply different population structure.

The mean population growth rates (u) can have spatial structure

$$\begin{bmatrix} u_{JF} \\ u_N \\ u_S \end{bmatrix}$$

unconstrained (all different)

$$\begin{bmatrix} u \\ u \\ u \end{bmatrix}$$

all the same

$$\begin{bmatrix} u_{JF} \\ u_{N\&S} \\ u_{N\&S} \end{bmatrix}$$

Strait of Juan de Fuca different
North and South same

The process error var-cov matrix can have structure: $\mathbf{e}_t \sim \mathrm{MVN}(0,\mathbf{Q})$

$$\begin{bmatrix} \sigma_{JF}^2 & \sigma_{JF,N} & \sigma_{JF,S} \\ \sigma_{JF,N} & \sigma_N^2 & \sigma_{N,S} \\ \sigma_{JF,S} & \sigma_{N,S} & \sigma_S^2 \end{bmatrix}$$

unconstrained

variances all different and year-to-year population changes covary

$$\begin{bmatrix} \sigma_{JF}^2 & 0 & 0 \\ 0 & \sigma_N^2 & 0 \\ 0 & 0 & \sigma_S^2 \end{bmatrix}$$

diagonal

unique variances and year-to-year population changes are uncorrelated

$$\begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$

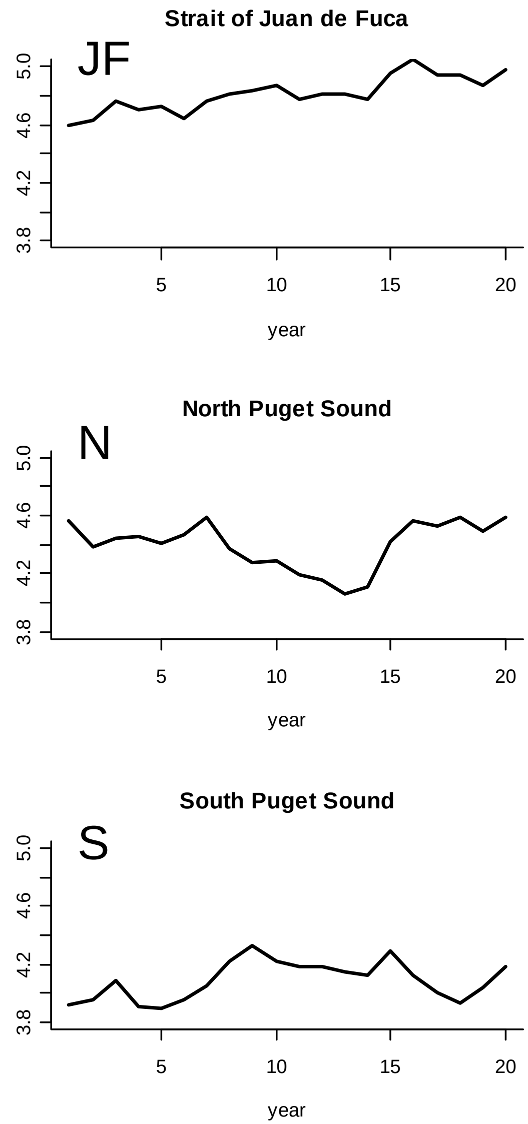
diagonal

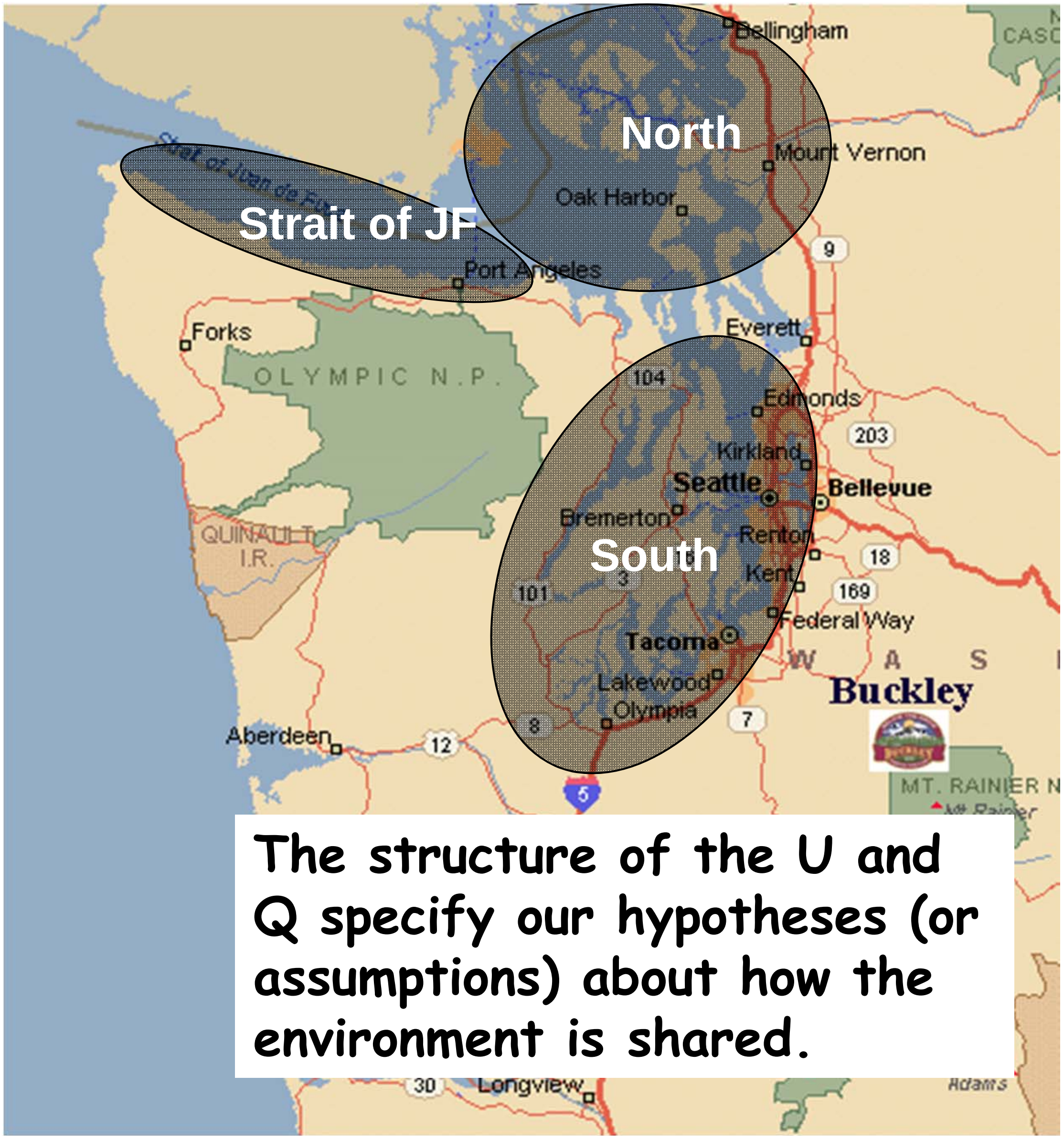
same variances and year-to-year population changes are uncorrelated

$$\begin{bmatrix} \sigma^2 & \alpha & \alpha \\ \alpha & \sigma^2 & \alpha \\ \alpha & \alpha & \sigma^2 \end{bmatrix}$$

JF has unique variance;
N & S share the same variance
yr-to-yr changes have equal covariance

$x_t = x_{t-1} + u + e_t$

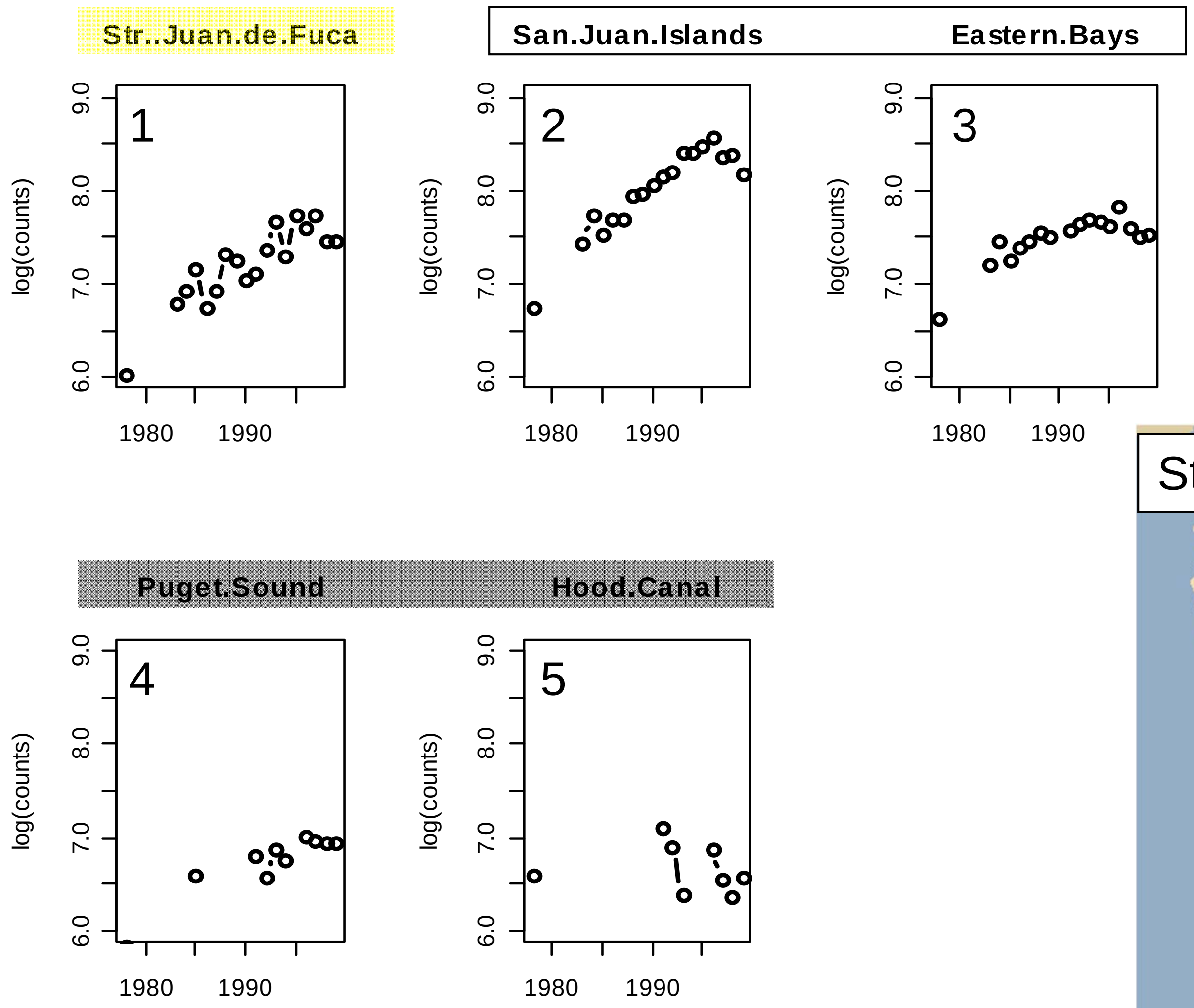




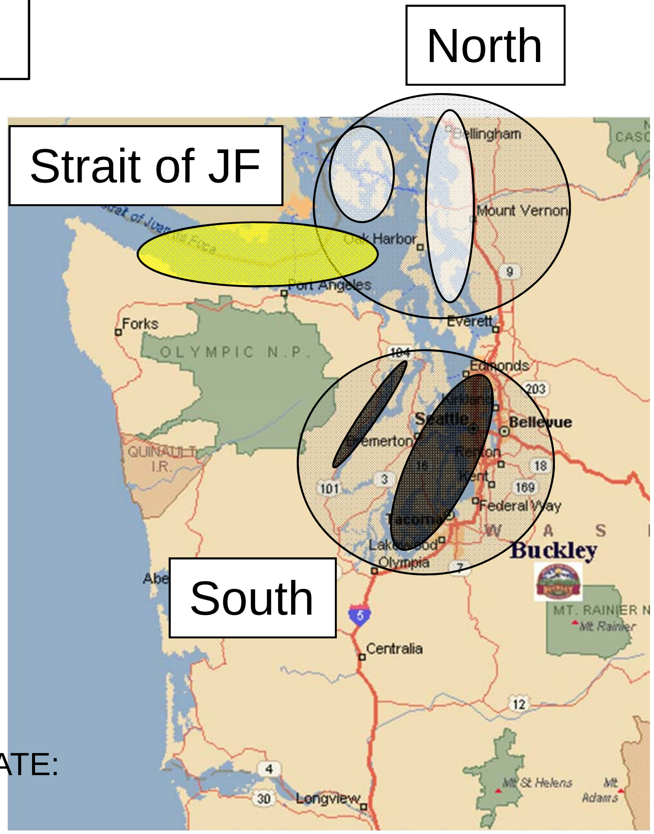
The structure of the U and Q specify our hypotheses (or assumptions) about how the environment is shared.

We observe x and those observations have error

The obs. err. model specifies how the observed time series are related to the true subpopulation sizes



5 sampling locations



Jeffries et al. 2003. TRENDS AND STATUS OF HARBOR SEALS IN WASHINGTON STATE: 1978–1999. J. Wildl. Manage. 67(1):208-219

The observation model

true population "hidden"

Log of counts

$$
\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \\ y_{5,t} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{ccc} JF & N & S \end{array} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
\times
\begin{bmatrix} x_{JF,t} \\ x_{N,t} \\ x_{S,t} \end{bmatrix}
+
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}
+
\begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \\ \eta_{3,t} \\ \eta_{4,t} \\ \eta_{5,t} \end{bmatrix}
$$

Z matrix

relates each observation time series to a different state process

observations

observation biases

measurement errors

The observation errors have a var-cov matrix

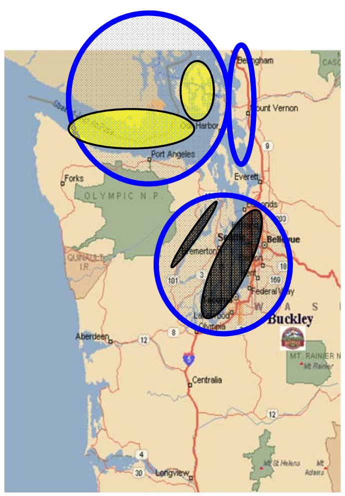
$$\begin{bmatrix} \eta_1^2 & \eta_{1,2} & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} \\ \eta_{1,2} & \eta_2^2 & \eta_{3,2} & \eta_{2,4} & \eta_{2,5} \\ \eta_{1,3} & \eta_{3,2} & \eta_3^2 & \eta_{3,4} & \eta_{3,5} \\ \eta_{1,4} & \eta_{2,4} & \eta_{3,4} & \eta_4^2 & \eta_{4,5} \\ \eta_{1,5} & \eta_{2,5} & \eta_{3,5} & \eta_{4,5} & \eta_5^2 \end{bmatrix} \quad \begin{bmatrix} \eta_1^2 & 0 & 0 & 0 & 0 \\ 0 & \eta_2^2 & 0 & 0 & 0 \\ 0 & 0 & \eta_3^2 & 0 & 0 \\ 0 & 0 & 0 & \eta_4^2 & 0 \\ 0 & 0 & 0 & 0 & \eta_5^2 \end{bmatrix} \quad \begin{bmatrix} \eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \end{bmatrix}$$

unconstrained	unique variances and uncorrelated errors	identical variances and uncorrelated errors
	diagonal	diagonal

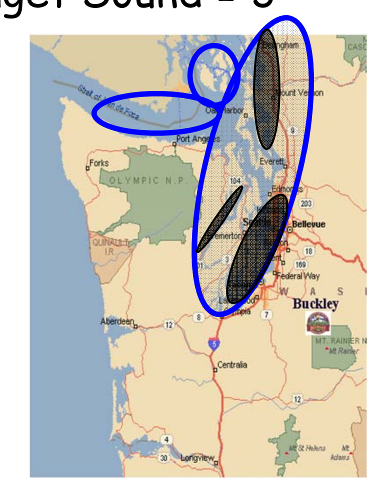
Instead of N, S, Str. J subpopulations, we could have other combinations and numbers of subpopulations

$$\begin{array}{r} \text{Str. JF} \\ \text{San Isl.} \\ \text{E. Bays} \\ \text{P.S.} \\ \text{Hood C.} \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Str of Juan de Fuca & San Juan Is sites = 1st subpop
Eastern bays = 2nd
Hood C. & S. Puget S. = 3rd



Strait of Juan de Fuca = 1st sub pop
San Juan Is sites = 2nd
Eastern bays, Hood Canal & S. Puget Sound = 3rd



One Puget Sound population and all sites are sampling it
One population

The harbor seal multivariate state-space model (MSSM) … in matrix form

3x1 vectors

3x3 matrix

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{u} + \mathbf{w}_t \quad \text{where } \mathbf{w}_t \sim MVN(0, \mathbf{Q})$$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{v}_t \quad \text{where } \mathbf{v}_t \sim MVN(0, \mathbf{R})$$

5x5 matrix

5x1 vectors

* We can add covariates that to explain some of the variability - Lecture 3

$$\mathbf{x}_t = \mathbf{x}_{t-1} + \mathbf{u} + \mathbf{C}\mathbf{c}_{t-1} + \mathbf{w}_t$$

$$\mathbf{y}_t = \mathbf{Z}\mathbf{x}_t + \mathbf{a} + \mathbf{D}\mathbf{d}_t + \mathbf{v}_t$$

c and d are covariates (like temperature) that you are using to explain some of the variability

$$\mathbf{C}\mathbf{c}_t = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{1,2} & c_{2,2} \\ c_{1,3} & c_{3,2} \end{bmatrix} \times \begin{bmatrix} c_{1,t} \\ c_{2,t} \end{bmatrix}$$

Computer lab

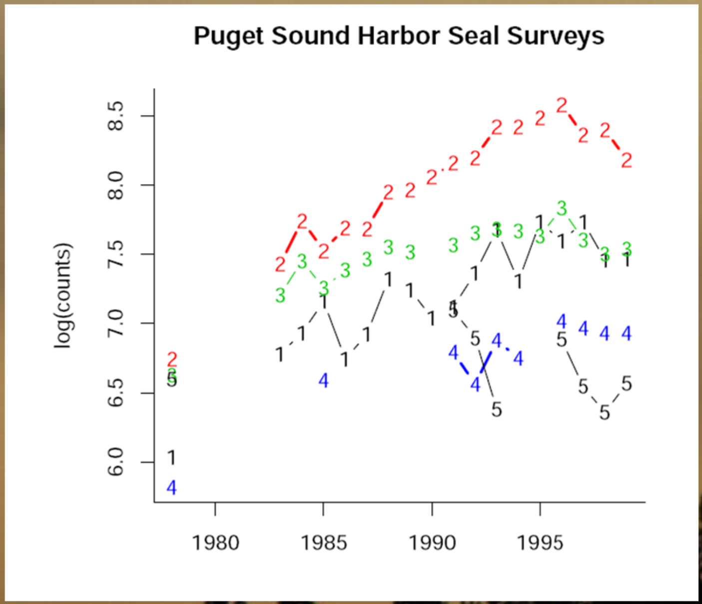
Chapter 7 & 8: Identifying spatial structure and covariance in harbor seals on the west coast of the USA

Place names you'll see in the code

Computer lab #1: Get used to multivariate modeling

Chapter 7: Combining multi-site data to estimate regional population trends

- RShowDoc("Chapter_SealTrend.R",package="MARSS")
- Work through through the chapter and text on your own.



Computer lab #2: Use model select to test hypotheses about subpop structure

Chapter 8: Identifying spatial structure and covariance in harbor seals on the west coast of the USA

- RShowDoc("Chapter_SealPopStructure.R",package="MARSS")
- Work through to section 8.4

2000km